

Laplace Transforms

For the design of a control system, it is important to know how the system of interest behaves and how it responds to different controller designs. To do this, the dynamic equations of the system are obtained and are solved to get the dynamic response. There are three different domains within which the dynamic response of a system is studied for the purpose of control design. These are the Laplace domain, the frequency domain and the state-space. This module provides an introduction to the Laplace domain and covers the mathematics of the Laplace transform.

with(inttrans) :

(This command loads the functions required for computing Laplace and Inverse Laplace transforms)

The Laplace transform

The Laplace transform is a mathematical tool that is commonly used to solve differential equations. Not only is it an excellent tool to solve differential equations, but it also helps in obtaining a qualitative understanding of how a system will behave and how changing certain parameters will effect the dynamic response.

Definitions

The Laplace transform of a real function $f(t)$ is defined as

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t) \cdot e^{-s \cdot t} dt, \quad t \geq 0$$

... Eq. (1)

where s is a complex variable ($s = \sigma + j \cdot \omega$). This definition is only valid if

$\int_0^{\infty} |f(t)| \cdot e^{-\sigma \cdot t} dt < \infty$ for some finite and real value of σ . The Laplace transform provides

the ability to transform a differential equation into a form that can be manipulated

algebraically. For example, the differential equation

$$A \cdot \ddot{f}(t) + B \cdot \dot{f}(t) + C \cdot f(t) = g(t)$$

... Eq. (2)

with initial conditions $\ddot{f}(0) = 0$ and $\dot{f}(0)$ can be transformed to

$$A \cdot s^2 \cdot F(s) + B \cdot s \cdot F(s) + C \cdot F(s) = G(s)$$

... Eq. (3)

which can be rearranged to obtain

$$F(s) = \frac{G(s)}{A \cdot s^2 + B \cdot s + C}$$

... Eq. (4)

The solution to the differential equation is then the inverse Laplace transform which is defined as

$$\mathcal{L}^{-1}[F(s)] = f(t) = \lim_{T \rightarrow \infty} \frac{1}{2 \cdot \pi \cdot j} \int_{c - T \cdot j}^{c + T \cdot j} F(s) \cdot e^{s \cdot t} ds$$

... Eq. (5)

where c is a constant that is greater than the real parts of all the singularities of $F(s)$ and j is $\sqrt{-1}$. This equation is usually not used in practice. Instead, the Laplace inverse for different functions can be found using tables like Table 1 (see below) that list functions and their transforms (these are easily available in textbooks and the internet). When the functions are not simple enough such that their transforms can be directly found from tables, the theorems of Laplace transforms and different rearrangement techniques (for example the partial fraction expansion) are used to arrange the equations in a form that can be recognized as one or a combination of the functions available in the tables. The inverse Laplace Transforms of functions can also be easily found using built-in functions

in Maple.

Example 1: Laplace transform of a unit step function

Find the Laplace transform of $f(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$.

Solution by hand

$$\begin{aligned}\mathcal{L}[f(t)] &= F(s) = \int_0^{\infty} f(t) \cdot e^{-s \cdot t} dt \\ &= -\frac{1}{s} \cdot e^{-s \cdot t} \Big|_0^{\infty} \\ &= \frac{1}{s}\end{aligned}$$

Solution using Maple

$$f := \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} ;$$

`laplace(f, t, s)`

$$\frac{1}{s}$$

(1.2.2.1)

Example 2: Laplace transform of a ramp function

Find the Laplace transform of $f(t) = A \cdot t$

where A is a constant.

Solution by hand

$$\begin{aligned}\mathcal{L}[f(t)] &= F(s) = \int_0^{\infty} f(t) \cdot e^{-s \cdot t} dt \\ &= \int_0^{\infty} A \cdot t \cdot e^{-s \cdot t} dt\end{aligned}$$

Integrating by parts ($\int_0^{\infty} u dv = u \cdot v \Big|_0^{\infty} - \int_0^{\infty} v du$):

$$\begin{aligned}
 F(s) &= -\frac{A \cdot t}{s} \cdot e^{-s \cdot t} \Big|_0^\infty - \int_0^\infty \frac{-A \cdot e^{-s \cdot t}}{s} dt \\
 &= \frac{A}{s} \cdot \left(-\frac{1}{s} \cdot e^{-s \cdot t} \right) \Big|_0^\infty \\
 &= \frac{A}{s^2}
 \end{aligned}$$

Solution using Maple

$$\text{laplace}(A \cdot t, t, s)$$

$$\frac{A}{s^2}$$

(1.3.2.1)

Example 3: Laplace transform of a derivative

Find the Laplace transform of $\frac{dx}{dt}$.

Solution by hand

$$\mathcal{L}\left[\frac{dx}{dt}\right] = \int_0^\infty \frac{dx}{dt} \cdot e^{-s \cdot t} dt$$

Integrating by parts ($\int_0^\infty u dv = u \cdot v \Big|_0^\infty - \int_0^\infty v du$):

$$= e^{-st} \cdot x \Big|_0^\infty - \int_0^\infty x \cdot (-s \cdot e^{-st}) dt$$

$$= 0 - x(0) + s \cdot X(s)$$

Therefore,

$$\mathcal{L}\left[\frac{dx}{dt}\right] = s \cdot X(s) - x(0)$$

where $X(s)$ is the Laplace transform of $x(t)$.

Solution with Maple

$$\text{laplace}(\dot{x}(t), t, s)$$

$$s \text{ laplace}(x(t), t, s) - x(0)$$

(1.4.2.1)

$$\text{eval}((1.4.2.1), [\text{laplace}(x(t), t, s) = X(s)])$$

$$s X(s) - x(0)$$

(1.4.2.2)

Example 4: Laplace transform of a second derivative

Find the Laplace transform of $\frac{d^2x}{dt^2}$.

Solution by hand

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = \int_0^{\infty} \frac{d^2x}{dt^2} \cdot e^{-s \cdot t} dt$$

Integrating by parts ($\int_0^{\infty} u dv = u \cdot v|_0^{\infty} - \int_0^{\infty} v du$):

$$= e^{-st} \cdot \frac{dx}{dt} \Big|_0^{\infty} - \int_0^{\infty} \frac{dx}{dt} \cdot (-s \cdot e^{-st}) dt$$

$$= 0 - \frac{d}{dt}x(0) - \int_0^{\infty} \frac{dx}{dt} \cdot (-s \cdot e^{-st}) dt$$

Using the result from Example 3, this can be written as

$$= 0 - \frac{d}{dt}x(0) + s \cdot (s \cdot X(s) - x(0))$$

Therefore,

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = s^2 \cdot X(s) - s \cdot x(0) - \frac{d}{dt}x(0)$$

Solution with Maple

$\text{laplace}(\ddot{x}(t), t, s)$

$$s^2 \text{laplace}(x(t), t, s) - D(x)(0) - s x(0) \tag{1.5.2.1}$$

$\text{eval}(\text{(1.5.2.1)}, [\text{laplace}(x(t), t, s) = X(s)])$

$$s^2 X(s) - D(x)(0) - s x(0) \tag{1.5.2.2}$$

The general equation for Laplace transforms of derivatives

From Examples 3 and 4 it can be seen that if the initial conditions are zero, then taking a derivative in the time domain is equivalent to multiplying by s in the Laplace domain. The following is the general equation for the Laplace transform of a derivative of order n .

$$\mathcal{L}^{-1}\left[\frac{d^n}{dt^n}f(t)\right] = s^n \cdot F(s) - s^{n-1} \cdot f(0) - s^{n-2} \cdot \frac{d}{dt}f(0) \dots - \frac{d^{n-1}}{dt^{n-1}}f(0)$$

... Eq. (6)

For example, the seventh order derivative of a function $x(t)$ can be written as

$$\text{laplace}\left(\frac{d^7}{dt^7}x(t), t, s\right)$$

$$s^7 \text{laplace}(x(t), t, s) - D^{(6)}(x)(0) - s D^{(5)}(x)(0) - s^2 D^{(4)}(x)(0) - s^3 D^{(3)}(x)(0) \quad (1.6.1)$$

$$- s^4 D^{(2)}(x)(0) - s^5 D(x)(0) - s^6 x(0)$$

eval(1.6.1), [laplace(x(t), t, s) = X(s)]

$$s^7 X(s) - D^{(6)}(x)(0) - s D^{(5)}(x)(0) - s^2 D^{(4)}(x)(0) - s^3 D^{(3)}(x)(0) - s^4 D^{(2)}(x)(0) \quad (1.6.2)$$

$$- s^5 D(x)(0) - s^6 x(0)$$

Example 5: Laplace transform of a sine function

Find the Laplace transform of $f(t) = A \cdot \sin(\omega \cdot t)$.

Solution by hand

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} A \cdot \sin(\omega \cdot t) \cdot e^{-s \cdot t} dt$$

$$\text{Using the identity } \sin(a) = \frac{1}{2 \cdot j} \cdot (e^{j \cdot a} - e^{-j \cdot a}),$$

$$F(s) = \int_0^{\infty} \frac{A}{2 \cdot j} \cdot (e^{j \cdot \omega \cdot t} - e^{-j \cdot \omega \cdot t}) \cdot e^{-s \cdot t} dt$$

$$= \frac{A}{2 \cdot j} \cdot \left[\frac{1}{s - j \cdot \omega} - \frac{1}{s + j \cdot \omega} \right]$$

$$= \frac{A \cdot \omega}{s^2 + \omega^2}$$

Solution using Maple

$$\text{laplace}(A \cdot \sin(\omega \cdot t), t, s)$$

$$\frac{A \omega}{\omega^2 + s^2}$$

(1.7.2.1)

List of Functions and their Laplace Transforms

The following table contains a list of some common functions and their Laplace transforms.

Table 1. Common functions and their Laplace transforms

Time domain function, $f(t)$	Laplace transform, $F(s)$
Unit impulse, $f(t) = \delta(t)$	1
Unit-step, $f(t) = 1$	$\frac{1}{s}$
Unit-ramp, $f(t) = t$	$\frac{1}{s^2}$
$f(t) = t^n, n > 0$	$\frac{n!}{s^{n+1}}$
$f(t) = e^{-a \cdot t}$	$\frac{1}{s + a}$
$f(t) = t \cdot e^{-a \cdot t}$	$\frac{1}{(s + a)^2}$
$f(t) = t^n \cdot e^{-a \cdot t}$	$\frac{n!}{(s + a)^{n+1}}$
$f(t) = \sin(\omega \cdot t)$	$\frac{\omega}{s^2 + \omega^2}$
$f(t) = \cos(\omega \cdot t)$	$\frac{s}{s^2 + \omega^2}$
$f(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} \cdot e^{-\zeta \cdot \omega_n \cdot t} \cdot \sin(\omega_n \cdot \sqrt{1 - \zeta^2} \cdot t)$ where $0 < \zeta < 1$	$\frac{\omega_n^2}{s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \omega_n^2}$
$f(t) = \frac{d}{d t} g(t)$	$s \cdot G(s) - g(0)$
$f(t) = \frac{d^n}{d t^n} h(t)$	$s^n \cdot H(s) - s^{n-1} \cdot h(0) - s^{n-2} \cdot \frac{d}{d t} h(0) \dots$ $- \frac{d^{n-1}}{d t^{n-1}} h(0)$

Important Properties and Theorems of Laplace Transforms

1. Multiplication by a constant

$$\mathcal{L}[k \cdot f(t)] = k \cdot F(s)$$

... Eq. (7)

where k is a constant.

2. Superposition

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

... Eq. (8)

3. Differentiation

$$\mathcal{L}^{-1}\left[\frac{d^n}{dt^n} f(t)\right] = s^n \cdot F(s) - s^{n-1} \cdot f(0) - s^{n-2} \cdot \frac{d}{dt} f(0) \dots - \frac{d^{n-1}}{dt^{n-1}} f(0)$$

... Eq. (9)

4. Integration

$$\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}$$

... Eq. (10)

5. Complex shifting

$$\mathcal{L}[e^{\mp \alpha t} \cdot f(t)] = F(s \pm \alpha)$$

... Eq. (11)

where α is a constant.

6. The Initial Value Theorem

$$f(0) = \lim_{s \rightarrow \infty} s \cdot F(s)$$

... Eq. (12)

7. The Final Value Theorem

$$f(\infty) = \lim_{s \rightarrow 0} s \cdot F(s)$$

... Eq. (13)

8. Time delay

$$\mathcal{L}[f(t - \tau)] = e^{-s \cdot \tau} \cdot F(s)$$

... Eq. (14)

where τ is the time delay.

9. Time Domain Convolution

$$\mathcal{L}\left[\int_0^t f_1(\tau) \cdot f_2(t - \tau) \, d\tau\right] = F_1(s) \cdot F_2(s) = \mathcal{L}\left[\int_0^t f_1(t - \tau) \cdot f_2(\tau) \, d\tau\right]$$

... Eq. (15)

Example 6: Laplace transform of $f(t) = t \cdot e^{2 \cdot t}$

Find the Laplace transform of $f(t) = t \cdot e^{2 \cdot t}$.

Solution by hand

Using Table 1,

$$\mathcal{L}[t] = F(s) = \frac{1}{s^2}$$

Therefore, using property 5. Complex shifting,

$$\mathcal{L}[t \cdot e^{2 \cdot t}] = F(s - 2) = \frac{1}{(s - 2)^2}$$

▼ **Solution using Maple**

$$\text{laplace}(t \cdot e^{2 \cdot t}, t, s)$$

$$\frac{1}{(s-2)^2}$$

(1.10.2.1)

▼ **Example 7: Laplace transform of $f(t) = 4 \cdot e^t \cdot (\cos(2 \cdot t) + 9)$**

Find the Laplace transform of $f(t) = 4 \cdot e^t \cdot (\cos(2 \cdot t) + 9)$.

▼ **Solution by hand**

The Laplace transform of this function can be found using Table 1 and Properties 1, 2 and 5.

$$\begin{aligned} \mathcal{L}[4 \cdot e^t \cdot (\cos(2 \cdot t) + 9)] &= \mathcal{L}[4 \cdot e^t \cdot \cos(2 \cdot t) + 36 \cdot e^t] \\ &= \frac{4 \cdot (s-1)}{(s-1)^2 + 4} + \frac{36}{s-1} \\ &= \frac{4 \cdot (s-1)}{5 - 2 \cdot s + s^2} + \frac{36}{s-1} \\ &= \frac{-40 \cdot s^2 + 80 \cdot s - 184}{-s^3 + 3 \cdot s^2 - 7 \cdot s + 5} \end{aligned}$$

▼ **Solution using Maple**

$$\text{laplace}(4 \cdot e^t \cdot (\cos(2 \cdot t) + 9), t, s)$$

$$\frac{2}{s-1-2I} + \frac{2}{s-1+2I} + \frac{36}{s-1}$$

(1.11.2.1)

simplify

$$\frac{-40s^2 + 80s - 184}{-s^3 + 3s^2 - 7s + 5}$$

(1.11.2.2)

▼ **Example 8: Laplace transform of $F(s) = \frac{1}{s^2 + 4 \cdot s + 13}$**

Find the inverse Laplace transform of $F(s) = \frac{1}{s^2 + 4 \cdot s + 13}$.

▼ **Solution by hand**

$F(s) = \frac{1}{s^2 + 4 \cdot s + 13}$ can be rearranged to get

$$F(s) = \frac{1}{(s + 2)^2 + 9}$$

This resembles the form of the Laplace transform of a sine function. Also, the $(s + 2)^2$ term hints towards complex shifting. Further rearrangement gives

$$F(s) = \frac{1}{3} \cdot \frac{3}{(s + 2)^2 + 3^2}$$

Using Properties 1 and 5, and Table 1, the inverse Laplace transform of $F(s)$ is

$$\mathcal{L}^{-1}\left[\frac{1}{3} \cdot \frac{3}{(s + 2)^2 + 3^2}\right] = \frac{1}{3} \cdot e^{-2 \cdot t} \cdot \sin(3 \cdot t)$$

▼ **Solution using Maple**

$$\text{invlaplace}\left(\frac{1}{s^2 + 4 \cdot s + 13}, s, t\right)$$

$$\frac{1}{3} e^{-2t} \sin(3 t)$$

(1.12.2.1)

▼ **Example 9: Inverse Laplace transform of $F(s) = \frac{32}{s^3 + 3 \cdot s^2 + 2 \cdot s}$ (Method of**

Partial Fraction Expansion)

Find the inverse Laplace transform of $F(s) = \frac{32}{s^3 + 3 \cdot s^2 + 2 \cdot s}$.

▼ **Solution by hand**

This example shows how to use the method of Partial Fraction Expansion when there are no repeated roots in the denominator.

The denominator of the function can be factored to get

$$F(s) = \frac{32}{s \cdot (s + 1) \cdot (s + 2)}$$

This can be written as a sum of partial fractions:

$$\frac{32}{s \cdot (s + 1) \cdot (s + 2)} = \frac{A_1}{s} + \frac{A_2}{s + 1} + \frac{A_3}{s + 2}$$

... Eq. (16)

The next step is to solve for A_1 , A_2 and A_3 . Multiplying both sides by s , gives

$$\frac{32}{(s + 1) \cdot (s + 2)} = A_1 + \frac{s \cdot A_2}{s + 1} + \frac{s \cdot A_3}{s + 2}$$

By setting $s = 0$, this equation simplifies to

$$\frac{32}{2} = A_1$$

Similarly, multiplying both sides of Eq. (16) by $(s + 1)$ and setting $s = -1$, gives

$$\frac{32}{-1} = A_2$$

and multiplying both sides of Eq. (16) by $(s + 2)$ and setting $s = -2$, gives

$$\frac{32}{2} = A_3$$

Now $F(s)$ can be written as

$$F(s) = \frac{16}{s} - \frac{32}{s + 1} + \frac{16}{s + 2}$$

Using Properties 1 and 5, and using Table 1, we get

$$f(t) = \mathcal{L}^{-1} \left[\frac{16}{s} - \frac{32}{s + 1} + \frac{16}{s + 2} \right] = 16 - 32 \cdot e^{-t} + 16 \cdot e^{-2 \cdot t}$$

▼ Solution with Maple

$$\text{invlaplace} \left(\frac{32}{s \cdot (s + 1) \cdot (s + 2)}, s, t \right) = -32 e^{-t} + 16 e^{-2t} + 16 \quad (1.13.2.1)$$

▼ **Example 10: Inverse Laplace transform of $F(s) = \frac{32}{s^3 + 4 \cdot s^2 + 4 \cdot s}$ (Method of**

Partial Fraction Expansion)

Find the inverse Laplace transform of

$$F(s) = \frac{32}{s^3 + 4 \cdot s^2 + 4 \cdot s}$$

▼ Solution by hand

This example shows how to use the method of Partial Fraction Expansion when there are repeated roots in the denominator.

The denominator of the function can be factored to get

$$F(s) = \frac{32}{s \cdot (s + 2)^2}$$

This can be written as a sum of partial fractions:

$$\frac{32}{s \cdot (s + 2)^2} = \frac{A_1}{s} + \frac{A_2}{s + 2} + \frac{A_3}{(s + 2)^2}$$

... Eq. (17)

The next step is to solve for A_1 , A_2 and A_3 . Multiplying both sides by s , gives

$$\frac{32}{(s + 2)^2} = A_1 + \frac{s \cdot A_2}{(s + 2)} + \frac{s \cdot A_3}{(s + 2)^2}$$

By setting $s = 0$, this simplifies to

$$\frac{32}{4} = A_1$$

Similarly, multiplying both sides of Eq. (17) by $(s + 2)^2$ gives

$$\frac{32}{s} = \frac{A_1 \cdot (s + 2)^2}{s} + A_2 \cdot (s + 2) + A_3$$

... Eq. (18)

and setting $s = -2$, gives

$$\frac{32}{-2} = A_3$$

Now, differentiating both sides of Eq. (18) with respect to s and then setting $s = -2$ gives

$$\frac{d}{ds} \left[\frac{32}{s} \right] = \frac{d}{ds} \left[\frac{A_1 \cdot (s + 2)^2}{s} \right] + A_2$$

This reduces to

$$-\frac{32}{4} = A_2$$

$F(s)$ can now be written as

$$F(s) = \frac{8}{s} - \frac{8}{(s+2)} - \frac{16}{(s+2)^2}$$

Using Properties 1 and 5, and using Table 1, we get

$$f(t) = \mathcal{L}^{-1} \left[\frac{8}{s} - \frac{8}{(s+2)} - \frac{16}{(s+2)^2} \right] = 8 - 8 \cdot e^{-2 \cdot t} - 16 \cdot t \cdot e^{-2 \cdot t}$$

Solution with Maple

$$\text{invlaplace} \left(\frac{32}{s^3 + 4 \cdot s^2 + 4 \cdot s}, s, t \right) = 8 - 8 e^{-2t} (2t + 1) \quad (1.14.2.1)$$

Example 11: Inverse Laplace transform of $F(s) = \frac{16}{s^3 + 2 \cdot s^2 + 2 \cdot s}$ (Method of Partial Fraction Expansion)

Find the inverse Laplace transform of $F(s) = \frac{16}{s^3 + 2 \cdot s^2 + 2 \cdot s}$.

Solution by hand

This example shows how to use the method of Partial Fraction Expansion when there are complex roots in the denominator.

The denominator of the function can be factored to get

$$F(s) = \frac{16}{s \cdot (s + 1 - j) \cdot (s + 1 + j)}$$

This can be written as a sum of partial fractions:

$$\frac{16}{s \cdot (s + 1 - j) \cdot (s + 1 + j)} = \frac{A_1}{s} + \frac{A_2}{s + 1 - j} + \frac{A_3}{s + 1 + j}$$

... Eq. (19)

Multiplying both sides by s and setting $s = 0$, gives

$$\frac{16}{2} = A_1$$

Similarly, multiplying both sides of Eq. (19) by $s + 1 - j$ and setting $s = -1 + j$, gives

$$\frac{16}{-2 - 2 \cdot j} = A_2$$

By rationalizing the denominator this can be written as

$$A_2 = -4 \cdot (1 - j)$$

Once again, multiplying both sides of Eq. (19) by $s + 1 + j$ and setting $s = -1 - j$, gives

$$\frac{16}{-2 + 2 \cdot j} = A_3$$

and by rationalizing the denominator this can be written as

$$A_3 = -4 \cdot (1 + j)$$

$F(s)$ can now be written as

$$F(s) = \frac{8}{s} - \frac{4 \cdot (1 - j)}{(s + 1 - j)} - \frac{4 \cdot (1 + j)}{(s + 1 + j)}$$

The inverse transform of this is

$$f(t) = 8 - 4 \cdot (1 - j) \cdot e^{(-1 + j) \cdot t} - 4 \cdot (1 + j) \cdot e^{(-1 - j) \cdot t}$$

which can be further written as

$$\begin{aligned} f(t) &= 8 - 4 \cdot e^{-t} \cdot [(1 - j) \cdot e^{j \cdot t} + (1 + j) \cdot e^{-j \cdot t}] \\ &= 8 - 4 \cdot e^{-t} \cdot [(e^{j \cdot t} + e^{-j \cdot t}) - j \cdot (e^{j \cdot t} - e^{-j \cdot t})] \\ &= 8 - 8 \cdot e^{-t} \cdot \left[\frac{(e^{j \cdot t} + e^{-j \cdot t})}{2} - \frac{j^2 \cdot (e^{j \cdot t} - e^{-j \cdot t})}{2 \cdot j} \right] \end{aligned}$$

Using the identities $\frac{(e^{j \cdot \omega \cdot t} + e^{-j \cdot \omega \cdot t})}{2} = \cos(\omega \cdot t)$ and $\frac{(e^{j \cdot \omega \cdot t} - e^{-j \cdot \omega \cdot t})}{2 \cdot j} = \sin(\omega \cdot t)$, we

get

$$f(t) = 8 - 8 \cdot e^{-t} \cdot [\cos(t) + \sin(t)]$$

▼ Solution with Maple

$$\text{invlaplace}\left(\frac{16}{s^3 + 2 \cdot s^2 + 2 \cdot s}, s, t\right)$$

$$8 - 8 e^{-t} (\cos(t) + \sin(t))$$

(1.15.2.1)

Example 12: Spring-mass system with viscous damping

Problem Statement: The following differential equation is the equation of motion for an ideal spring-mass system with damping and an external force $F(t)$.

$$m \cdot \ddot{x}(t) + c \cdot \dot{x}(t) + k \cdot x(t) = F(t)$$

If $m = 1$ Kg, $c = 5$ N·s/m, $k = 4$ N/m, $F(t) = 2$ N, and $\dot{x}(0) = x(0) = 0$, find the solution of this differential equation using Laplace transforms.

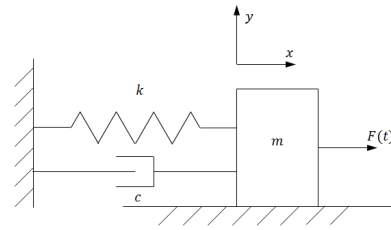


Fig. 1: Spring-mass system with damping

Solution

Taking the Laplace transform of both sides of the equation of motion gives

$$1 \cdot s^2 \cdot X(s) + 5 \cdot s \cdot X(s) + 4 \cdot X(s) = \frac{2}{s}$$

By rearranging this equation we get

$$X(s) = \frac{2}{s \cdot (s^2 + 5 \cdot s + 4)}$$

The denominator of this transfer function can be factorized to

$$X(s) = \frac{2}{s \cdot (s + 4) \cdot (s + 1)}$$

This can be further written as a sum of partial fractions:

$$\frac{2}{s \cdot (s + 4) \cdot (s + 1)} = \frac{A_1}{s} + \frac{A_2}{s + 4} + \frac{A_3}{s + 1}$$

... Eq. (20)

Now we have to solve for A_1 , A_2 and A_3 . Multiplying both sides by s , gives

$$\frac{2}{(s + 4) \cdot (s + 1)} = A_1 + \frac{s \cdot A_2}{s + 4} + \frac{s \cdot A_3}{s + 1}$$

By setting $s = 0$, this equation simplifies to

$$\frac{2}{4} = A_1$$

Similarly, multiplying both sides of Eq. (20) by $(s + 4)$ and setting $s = -4$, gives

$$\frac{2}{12} = A_2$$

and multiplying both sides of Eq. (20) by $(s + 1)$ and setting $s = -1$, gives

$$\frac{2}{-3} = A_3$$

Now $F(s)$ can be written as

$$X(s) = \frac{1}{2s} + \frac{1}{6 \cdot (s + 4)} - \frac{2}{3 \cdot (s + 1)}$$

Using Properties 1 and 5, and using Table 1, we get

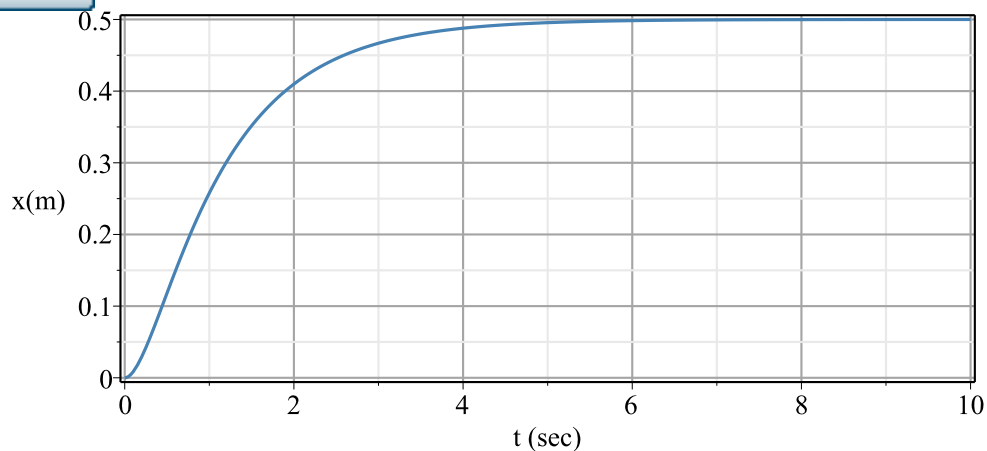
$$x(t) = \mathcal{L}^{-1} \left[\frac{1}{2s} + \frac{1}{6 \cdot (s + 4)} - \frac{2}{3 \cdot (s + 1)} \right] = \frac{1}{2} + \frac{e^{-4 \cdot t}}{6} - \frac{2}{3} \cdot e^{-t}$$

This equation shows that the displacement approaches $\frac{1}{2}$ m as t approaches infinity.

The following is a plot of the displacement versus time.



Displacement plot



Solution with Maple

$$\text{invlaplace}\left(\frac{2}{s \cdot (s^2 + 5 \cdot s + 4)}, s, t\right)$$

$$\frac{1}{2} + \frac{1}{6} e^{-4t} - \frac{2}{3} e^{-t}$$

(2.1.1.1)



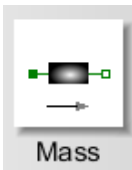
MapleSim Simulation

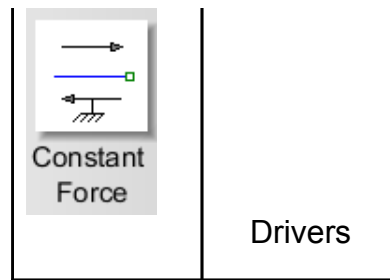
Constructing the model

Step1: Insert Components

Drag the following components into the workspace:

Table 2: Components and locations

Component	Location
 Translational Fixed	1-D Mechanical > Translational > Common
 Translational Spring Damper	1-D Mechanical > Translational > Common
 Mass	1-D Mechanical > Translational > Common
	1-D Mechanical > Translational > Force



Step 2: Connect the components

Connect the components as shown in the following diagram.

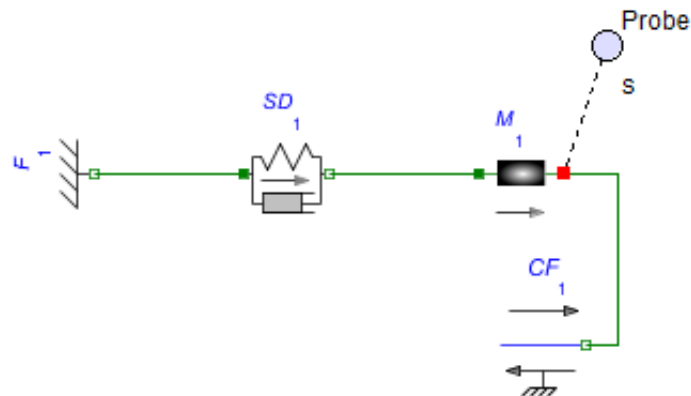


Fig. 2: Component diagram

Step 3: Set parameters and initial conditions

1. Click the **Translational Spring Damper** component and enter **4 N/m** for the spring constant (c) and **5 N/m** for the damping constant (d).
2. Click the **Mass** component and enter **1 kg** for the mass (m), **0 m/s** for the initial velocity (v_0) and **0 m** for the initial position (s_0). Select the check marks that enforce these initial condition.
3. Click the **Translational Constant Force** component and enter **2 N** for the **Nominal force** ($f_{constant}$).

Step 4: Run the Simulation

1. Attach a **Probe** to the **Mass** component as shown in Fig. 2. Click this probe and select **Length** in the **Inspector** tab. This will show the position of the mass as a

function of time.

2. Click **Run Simulation** (▶).

This simulation outputs the same plot that is obtained analytically:

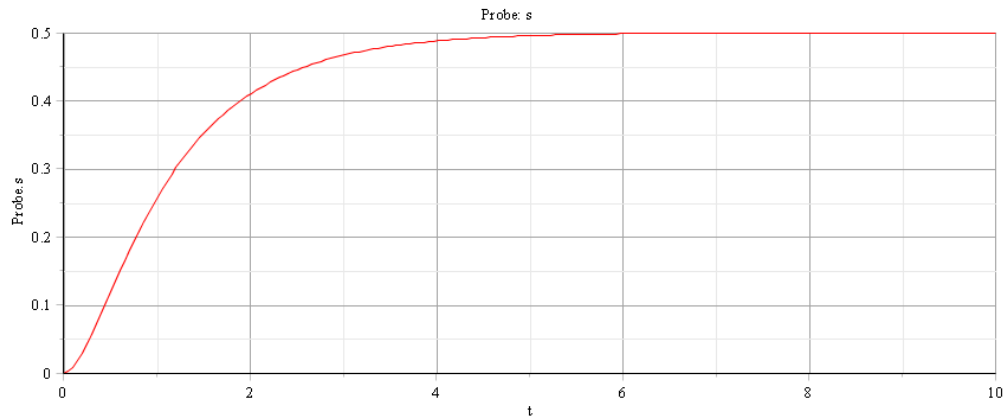


Fig. 3: MapleSim simulation result - Displacement (m) vs. time (sec)

References:

1. G.F. Franklin et al. "Feedback Control of Dynamic Systems", 5th Edition. Upper Saddle River, NJ, 2006, Pearson Education, Inc.
2. D. J. Inman. "Engineering Vibration", 3rd Edition. Upper Saddle River, NJ, 2008, Pearson Education, Inc.